There is the following theorem, ~~describ~~ giving an explicit expression of the Wronskian:

Theorem (Abel's Theorem): If y_1 and y_2 are solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)\cdot y = 0, \text{ where}$$

p and q are continuous on an open interval I, then the Wronskian is given by

$$W(y_1, y_2)(t) = c\exp\left(-\int p(t)\,dt\right).$$

Thus the Wronskian is always zero or never zero.

Proof:
$$\begin{cases} y_1'' + p(t)y_1' + q(t)y_1 = 0. \\ y_2'' + p(t)\cdot y_2' + q(t)\cdot y_2 = 0. \end{cases}$$

$$-y_2(y_1'' + p(t)y_1' + q(t)y_1)$$

$$+ y_1(y_2'' + p(t)y_2' + q(t)y_2)$$

$$= (-y_2y_1'' + y_1y_2'') + p(t)\cdot(y_1y_2' - y_1'y_2)$$

$$= W'(t) + p(t)\cdot W(t) = 0.$$

#

$W(t) = y_1y_2' - y_2'y_1'$

$W'(t) = y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1''$

$= y_1y_2'' - y_2y_1''$

Then there is the following theorem:

Theorem: Let y_1, y_2 be two solutions of the equations

$$y'' + p(t)y' + q(t)\cdot y = 0,$$

then the following statements are equivalent:

1). The functions y_1, y_2 are a fundamental set of solutions

2). The functions y_1, y_2 are linearly independent on I.

3). $W(y_1, y_2)(t_0) \neq 0$ for some t_0

4) $W(y_1, y_2)(t) \neq 0$ for all t.

Section 3.4 Complex Roots ~~and~~ of the characteristic Equation:

We continue the discussion of the equation

$$ay'' + by' + cy = 0, \text{ where } a, b, c \text{ are } \text{real numbers}$$

We were trying to find the solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0.$$

Suppose now that the roots of the characteristic equation are conjugate complex numbers $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$.

Then the expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \quad y_2(t) = \exp[(\lambda - i\mu)t]$$

Now ~~we~~ need to apply Euler's formula:

$$e^{it} = \cos t + i\sin t.$$

to get $$e^{(\lambda+i\mu)t} = e^{\lambda t}\cdot e^{i\mu t},$$
$$= e^{\lambda t}\cos\mu t + i\cdot e^{\lambda t}\sin\mu t.$$

Now unfortunately, the ~~sol~~ functions $y_1(t) = \exp[(\lambda+i\mu)t]$, $y_2(t) = \exp[(\lambda-i\mu)t]$ are complex valued. We want to find real ~~sol~~ valued solutions.

There is the following

$$y_1(t) + y_2(t) = e^{\lambda t}(\cos\mu t + i\sin\mu t)$$
$$+ e^{\lambda t}(\cos\mu t - i\sin\mu t)$$

$$= 2e^{\lambda t}\cos\mu t$$

and $$\frac{y_1(t) - y_2(t)}{i} = \frac{1}{i}\left[e^{\lambda t}(\cos\mu t + i\sin\mu t) - e^{\lambda t}(\cos\mu t - i\sin\mu t)\right]$$

$$= 2e^{\lambda t}\sin\mu t.$$

Thus we can obtain a pair of real-valued solutions:

$$u(t) = e^{\lambda t}\cos\mu t, \quad v(t) = e^{\lambda t}\sin\mu t.$$

It's straightforward to compute the Wronskian:

$$W(u,v)(t) = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix}$$

$= \mu \cdot e^{2\lambda t} \neq 0$ as long as $\mu \neq 0$. (complex roots)

So the general solution is given by

$$y = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t.$$

Example: Find the general solution of $y'' + y' + y = 0$.

The characteristic equation is

$$r^2 + r + 1 = 0.$$

$$r = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\cdot i}{2}.$$

$\Rightarrow \lambda = -\frac{1}{2}, \ \mu = \frac{\sqrt{3}}{2}.$

general solution is

$$y = c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

Section 3.5 Repeated roots; reduction of order.

Now we consider the case when the characteristic equation

$ar^2+br+c=0$ has repeated roots.

Apparently, the roots are $r_1=r_2=-\frac{b}{2a}$. (since $b^2-4ac=0$).

We can only get one solution $y_1(t)=e^{-\frac{bt}{2a}}$ by solving the characteristic equation.

Example: Solve the equation $y''+4y'+4=0$.

The characteristic equation is $r^2+4r+4=0 \Rightarrow r=-2$.

One solution is $y_1(t)=e^{-2t}$. ~~We~~

The idea to find the second solution is the following: we know that for any constant v, $v\cdot e^{-2t}$ is an ~~equ~~ solution. Now we generalize this a little: assume that a ~~fu~~ second solution is $v(t)\cdot e^{-2t}$. Let's plug it into the equation:

$$\left(v(t)\cdot e^{-2t}\right)''+4\cdot\left(v(t)\cdot e^{-2t}\right)'+4v(t)e^{-2t}$$

$$=\left(v'(t)e^{-2t}-2v(t)e^{-2t}\right)'+4\left(v'(t)e^{-2t}-2v(t)e^{-2t}\right)+4v(t)e^{-2t}$$

$$=v''(t)e^{-2t}-2v'(t)e^{-2t}-2v'(t)e^{-2t}+4v(t)e^{-2t}+4v'(t)e^{-2t}-8v(t)e^{-2t}$$
$$+4v(t)e^{-2t}=0$$

$\Rightarrow$. $v''(t)=0$. $\Rightarrow v'(t)=c_1 \Rightarrow v(t)=c_1t+c_2$.

where c_1, c_2 are arbitrary constants.

Thus ~~$y_2(t)=$~~ $c_1te^{-2t}+c_2e^{-2t}$ is a solution.

We can let $y_2(t)=t\exp(-2t)$.

We can also calcute the Wronskian:

$$W(y_1,y_2)(t)=\begin{vmatrix} e^{-2t} & te^{-2t} \\ (-2)e^{-2t} & e^{-2t}-2te^{-2t} \end{vmatrix}$$

$$= e^{-4t} \neq 0.$$

i.e., $y_1(t)=e^{-2t}$, $y_2(t)=te^{-2t}$ form a fundamental solution of the equation.

We can generalize this: suppose the equation $ay''+by'+cy=0$ ~~has~~ has characteristic equation w/ repeated real roots:

$r_1=r_2=-\frac{b}{2a}$, then the general solution is of the form $y(t)=c_1te^{-\frac{b}{2a}t}+c_2e^{-\frac{b}{2a}t}$.

Reduction of order: The procedure is actually more generally applicable.

Suppose we know one solution $y_1(t)$, not everywhere zero of

$$y'' + p(t)y' + q(t)y = 0.$$

To find a second solution, let

$$y = v(t)\, y_1(t),$$

Then $y' = v'(t)\, y_1(t) + v(t)\, y_1'(t)$.

$$y'' = v''(t)\, y_1(t) + 2v'(t)\, y_1'(t) + v(t)\, y_1''(t).$$

Then $y'' + p(t)\cdot y' + q(t)\, y$

$$= v''(t)\, y_1(t) + v'(t)\left(2y_1'(t) + p(t)\, y_1(t)\right)$$

$$+ v(t)\underbrace{\left(y_1''(t) + p(t)\, y_1'(t) + q(t)\, y_1(t)\right)}_{0}$$

$$= v''\cdot y_1 + v'\cdot(2y_1' + py_1)$$

This is a <u>1st order</u> linear equation ~~in~~ ~~of~~ for the function v', we know how to solve it.

Example: Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0,$$

find a second linearly independent solution.

We set $y = t^{-1} \cdot v(t)$, then

$$y' = \frac{v'(t)}{t} - \frac{v(t)}{t^2}, \quad y'' = v'' \cdot t^{-1} - 2v' t^{-2} + 2v \cdot t^{-3}.$$

$$2t^2 y'' + 3t y' - y =$$

$$= 2t^2 \left(v'' t^{-1} - 2v' t^{-2} + 2v \cdot t^{-3} \right) + 3t \left(v' \cdot t^{-1} - v \cdot t^{-2} \right)$$

$$- v' \cdot t^{-1} \cancel{+ v \cdot t^{-2}}$$

$$= 2t v'' - 4v' + 3 \cdot v' \cancel{- t^{-1} v}$$

$$= 2t v'' - \cancel{(t + \tfrac{1}{t}) v'} \, v' = 0.$$

$$2t (v')' = v'$$

$$\Rightarrow v'(t) = c \cdot t^{\frac{1}{2}}$$

$$\Rightarrow v(t) = \frac{2}{3} c t^{\frac{3}{2}} + k.$$

We can let $y_2(t) = t^{\frac{3}{2}} \cdot t^{-1} = t^{\frac{1}{2}}$.

Section 3.6 : Non-homogeneous equations :

We now consider the non-homogeneous equation

$$L[y] = y'' + p(t)\,y' + q(t)\cdot y = g(t), \quad (*)$$

where p, q are continuous functions on the open interval I.

The homogeneous equation

$$\cancel{L[y] =}\ y'' + p(t)\,y' + q(t)\cdot y = 0$$ is called the homogeneous equation corresponding to $(*)$.

Theorem: If Y_1 and Y_2 are two solutions of the nonhomogeneous equation $(*)$, then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation. More precisely,

$$Y_1 - Y_2 = c_1 y_1(t) + c_2 y_2(t) \text{ where}$$

$y_1(t)$, $y_2(t)$ are a fundamental set of solutions of homogeneous equ

Proof: Almost trivial: $L[Y_1](t) = g(t)$.

$$L[Y_2](t) = g(t)$$

$$\Rightarrow L[Y_1 - Y_2](t) = L[Y_1](t) - L[Y_2](t) = 0.$$

We can then conclude

Theorem: The general solution of the nonhomogeneous equation can be written in the form

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t),$$ where $Y(t)$ is some specific solution of the non-homogeneous equation.

In other words, to find the general solutions of the non-homogeneous equation, we need to do the following:

1). Find the general solution $c_1y_1(t)+c_2y_2(t)$ of the corresponding homogeneous equation

2). Find ~~the~~ one solution $Y(t)$ of the nonhomogeneous equation.

This is sometimes called a particular solution.

3). Add these functions together.

~~Now, we~~ In the rest of ~~these~~ this section, we focus on finding a particular solution $Y(t)$. The method is called

Method of Undetermined Coefficients:

Before examples, we say a little bit about this method: we have to make some initial assumption about the form of the particular solution $Y(t)$, but with coefficients unspecified.

Example 1: Find a particular solution of

$$y''-3y'-4y=3e^{2t},$$

We can see that the most reasonable guess of a particular solution is that

$$Y(t)=A\cdot e^{2t},$$

(A is the coefficient to be determined).

Then, $Y'(t) = 2Ae^{2t}$, $Y''(t) = 4Ae^{2t}$.

$$Y''(t) - 3Y'(t) - 4Y(t) = (4A - 6A - 4A)3e^{2t}.$$

$$\Rightarrow \quad A = -\frac{1}{2}.$$

Thus a particular solution is $Y(t) = -\frac{1}{2}e^{2t}$.

Example 2: Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t.$$

The first guess might be $y = A\sin t$. We can then easily see that this assumption does not work. A second guess is $y = A\sin t + B\cos t$.

then $y' = A\cos t - B\sin t$, and $y'' = -A\sin t - B\cos t$

Then
$$y'' - 3y' - 4y = -A\sin t - B\cos t - 3A\cos t + 3B\sin t - 4A\sin t - 4B\cos t$$
$$= (-5A + 3B)\sin t + (-5B - 3A)\cos t$$

$$\Rightarrow \begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0 \end{cases} \Rightarrow \quad A = -\frac{5}{17}, \quad B = -\frac{3}{17}$$

$$Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

Example 3: Find a particular solution of $y''-3y'-4y=-8e^{t}\cos 2t$

We make the assumption $Y(t)=Ae^{t}\cancel{\cos 2t}\cos 2t+Be^{t}\sin 2t$.

$$
\begin{aligned}
Y'(t) &= Ae^{t}\cos 2t-2Ae^{t}\sin 2t \\
&\quad + Be^{t}\sin 2t+2Be^{t}\cos 2t \\
&= (A+2B)e^{t}\cos 2t+(B-2A)e^{t}\sin 2t
\end{aligned}
$$

$$
\begin{aligned}
Y''(t) &= \big(A+2B+2(B-2A)\big)e^{t}\cos 2t \\
&\quad + \big(B-2A-2(A+2B)\big)e^{t}\sin 2t \\
&= (-3A+4B)e^{t}\cos 2t+(-3B-4A)e^{t}\sin 2t
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ & Y''(t)-3Y'(t)-4Y(t) \\
&= (-3A+4B-3A-6B-4A)e^{t}\cos 2t \\
&\quad + (-3B-4A-3B+6A-4B)e^{t}\sin 2t \\
&= (-10A-2B)e^{t}\cos 2t+(+2A-10B)e^{t}\sin 2t \\
&= -8e^{t}\cos 2t
\end{aligned}
$$

$$
\Rightarrow \begin{cases} 10A+2B=8 \\ 2A-10B=0 \end{cases} \Rightarrow A=\frac{10}{13},\ B=\frac{2}{13}
$$

Example 4: Find a particular solution of

$$y''-3y'-4y=2e^{-t}.$$

If we let ~~A(t) =~~ $Y(t)=Ae^{-t}$, then we see that

$$Y'(t)=-A\cdot e^{-t},\ Y''(t)=A\cdot e^{-t}.$$

$$\Rightarrow Y''-3Y'-4= \cancel{Ae^{-t}} (A+3A-4A)e^{-t}=0.$$

so it does not work.

~~Think above the~~

A second guess is $Y(t)=At\cdot e^{-t}$.

then $Y'(t)=A\cdot e^{-t}-Ate^{-t}$.

$$Y''(t)=-Ae^{-t}-Ae^{-t}+Ate^{-t}$$

$$=-2Ae^{-t}+Ate^{-t}$$

$$Y''(t)-3Y'(t)-4Y(t)$$

$$=-2Ae^{-t}+Ate^{-t}-3Ae^{-t}+3Ate^{-t}$$

$$-4Ate^{-t}$$

$$=-5Ae^{-t}$$

$$\Rightarrow A=-\frac{2}{5}.\qquad \text{And } Y(t)=-\frac{2}{5}te^{-t}$$

We can ~~generalize~~ summarize the steps involved in finding ~~the~~ a particular solution of ~~an~~

$$ay'' + by' + cy = g(t).$$

1. Make sure that the function $g(t)$ ~~is~~ belongs to the class of functions discussed: It involves nothing more than exponential functions, sines, cosines, polynomials or sums, products of these functions.

2. If $g(t) = g_1(t) + \cdots + g_n(t)$, ~~where~~ where g_i's are of the above form, we can split the problem to n subproblems: we only need to find Y_i's s.t.

$$aY_i'' + bY_i' + cY_i = g_i(t)$$

then $Y = \sum_{i=1}^{n} Y_i$ is a particular solution.

3. If the Y_i ~~we~~ assumed ~~happens~~ to be a solution of the corresponding homogeneous solution, then we need to multiply it by t (or even t^2 if necessary).

We conclude w/ the following table:

The Particular Solution of $ay''+by'+cy=g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t)=a_0t^n+a_1t^{n-1}+\cdots+a_n$	$t^s(A_0t^n+A_1t^{n-1}+\cdots+A_n)$
$P_n(t)=e^{\alpha t}$	$t^s(A_0t^n+A_1t^{n-1}+\cdots+A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s[(A_0t^n+\cdots+A_n)e^{\alpha t}\cos\beta t + (B_0t^n+\cdots+B_n)e^{\alpha t}\sin\beta t]$

Here $s=0,1,2$ is to make sure no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation.

Proof: We only prove the first case: $P_n(t)=a_0t^n+a_1t^{n-1}+\cdots+a_n$

The non-homogeneous equation is

$$ay''+by'+cy=a_0t^n+a_1t^{n-1}+\cdots+a_n$$

Assume that $Y(t)=A_0t^n+A_1t^{n-1}+\cdots+A_{n-1}t+A_n$

$$Y'(t)=nA_0t^{n-1}+(n-1)A_1t^{n-2}+\cdots+\cancel{A_{n-1}}A_{n-1}$$

$$Y''(t)=n(n-1)A_0t^{n-2}+(n-1)(n-2)A_1t^{n-3}+\cdots+2A_{n-2}$$

$\Rightarrow aY''(t)+bY'(t)+cY(t)$

$\cancel{=}$ $\cancel{a[A_0t^n+A_1t^{n-1}+\cdots+A_{n-1}t+A_n]}$

$=a[n(n-1)A_0t^{n-2}+\cdots+2A_{n-2}]+b[nA_0t^{n-1}+\cdots+A_{n-1}]$

$+c[A_0t^n+A_1t^{n-1}+\cdots+A_{n-1}t+A_n]=a_0t^n+\cdots+a_n$

$\Rightarrow$ By comparing coefficients of t^k:

$$c \cdot A_0 = a_0 \qquad (\text{coefficient of } t^n)$$

$$cA_1 + nbA_0 = a_1 \qquad (\qquad \text{of } t^{n-1})$$

$$\vdots \qquad\qquad (\text{coeffic. of } t^0)$$

$$cA_n + bA_{n-1} + 2aA_{n-2} = a_n$$

This equation has a solution if $c \neq 0$. (Some linear algebra)

3.6. Variation of parameters:

We now describe another method of finding a particular solution of a non-homogen equation, called variation of parameters. This is a general method in the following sense. It ~~can~~ does not require detailed assumptions about the form of the solution. We start from an example.

Example: Find a particular solution of

$$y'' + 4y = 3\csc t \left(= \frac{3}{\sin t}\right) \quad (*)$$

Solution: We can see that the method of undetermined coefficie does not apply here since it involve a quotine of $\sin t$. The homogeneous equation corresponding to $(*)$ is

$$y'' + 4y = 0$$

And the general solution is

$$y_c(t) = C_1 \cos 2t + C_2 \sin 2t.$$

The basic idea in the method of variation of parameters is to replace the constants C_1 and C_2 by functions $u_1(t)$ and $u_2(t)$, and to determine these functions to get a solution of $(*)$. i.e. $y = u_1(t)\cos 2t + u_2(t)\sin 2t$.

To determine $u_1(t)$ and $u_2(t)$, we can ~~plug to~~ plug this to equation $(*)$. ~~At~~ And ~~we~~ it is easy to see that we can get one equation about $u_1(t)$ and $u_2(t)$.

It is ~~easy~~ natual guess to ~~see~~ that ~~there~~ we need two equations if we have two indeterminates.

So we can impose a second condition. And ~~it~~ this can also simplify ~~the~~ ~~equa~~ computation:

$$y' = -2u_1(t)\sin 2t + u_1'(t)\cos 2t + 2u_2(t)\cos 2t + u_2'(t)\sin 2t.$$

Now, we impose the condition $u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$.

(Explain the reason why this simplifies the equation).

Then $y' = 2u_2(t)\cos 2t - 2u_1(t)\sin 2t$,

and
$$y'' = -4u_2(t)\sin 2t + 2u_2'(t)\cos 2t$$
$$-4u_1(t)\cos 2t - 2u_1'(t)\sin 2t$$

Recall that the original equation is

$$y'' + 4y = 3\csc t.$$

$$\Downarrow$$

$$-4u_2(t)\sin 2t - 4u_1(t)\cos 2t + 2u_2'(t)\cos 2t - 2u_1'(t)\sin 2t$$
$$+ 4u_1(t)\cos 2t + 4u_2(t)\sin 2t = 2u_2'(t)\cos 2t - 2u_1'(t)\sin 2t$$
$$= 3\csc t.$$

So finally, we get two equations: (about u_1 and u_2).

$$\begin{cases} u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0. & (1) \\ 2u_2'(t)\cos 2t - 2u_1'(t)\sin 2t = 3\csc t & (2). \end{cases}$$

From (1) we get $u_2'(t) = -u_1'(t)\dfrac{\cos 2t}{\sin 2t}$,

Then plug this into (2), we obtain:

$$-2u_1'(t)\frac{\cos^2 2t}{\sin 2t} - 2u_1' \sin 2t = 3\csc t.$$

$$(-2)\cdot u_1'(t)\frac{1}{\sin 2t} = 3\csc t.$$

$$u_1'(t) = -\frac{3}{2}\cdot\frac{\sin 2t}{\sin t} = -\frac{3}{2}\cdot\frac{2\sin t\cos t}{\sin t}.$$

$$= -3\cos t,$$

$$\Rightarrow u_1(t) = -3\sin t + C_1$$

And $u_2'(t) = 3\cdot\dfrac{\cos t\cos 2t}{\sin 2t} = \dfrac{3}{2}\cdot\dfrac{\cancel{\cos t}\cos 2t}{\sin t\,\cancel{\cos t}}$

$$= \frac{3}{2}\,\frac{1-2\sin^2 t}{\sin t}$$

$$= \frac{3}{2} \csc t - 3 \sin t$$

Integrate this, we get

$$u_2(t) = \frac{3}{2} \ln \left| \csc t - \cot t \right| + 3 \cos t.$$

Let us generalize this method:

The equation $y'' + p(t) y' + q(t) y = g(t)$. $(*)$

Where p, q, g are given continuous functions.

~~Suppose now~~ The corresponding homogeneous equation

$y'' + p(t) y' + q(t) y = 0$ has a fundamental set of equations $y_1(t)$ and $y_2(t)$.

We now ~~replace it~~ try to find a particular solution of equation $(*)$ of the form.

$$y = u_1(t) y_1(t) + u_2(t) \cdot y_2(t).$$

Now $y' = u_1'(t) y_1(t) + u_2'(t) \cdot y_2(t) + u_1(t) y_1'(t) + u_2(t) y_2'(t)$.

We now impose the condition

$$u_1'(t) y_1(t) + u_2'(t) y_2(t) = 0.$$

And $y' = u_1(t) y_1'(t) + u_2(t) y_2'(t)$.

5.

And $y'' = u_1'(t)\cdot y_1'(t) + u_1(t)\, y_1''(t)$

$+ u_2'(t)\, y_2'(t) + u_2(t)\cdot y_2''(t)$.

Then the original equa. reads:

$$u_1'(t)\cdot y_1'(t) + \underline{u_1(t)\, y_1''(t)} + u_2'(t)\, y_2'(t) + \underline{\underline{u_2(t)\cdot y_2''(t)}}$$

$$+ \underline{p(t)\cdot u_1(t)\cdot y_1'(t)} + \underline{\underline{p(t)\, u_2(t)\cdot y_2'(t)}} + \underline{q(t)\cdot u_1(t)\, y_1(t)} + \underline{\underline{q(t)\, u_2(t)\, y_2(t)}}$$

$$= g(t).$$

$$\Rightarrow \begin{cases} u_1'(t)\, y_1'(t) + \cancel{u_1(t)\, y_1''(t)}\; u_2'(t)\, y_2'(t) = g(t). & (1) \\ u_1'(t)\cdot y_1(t) + u_2'(t)\cdot y_2(t) = 0. & (2) \end{cases}$$

~~$u_1'(t)\cdot v$~~

$(1)\cdot y_2(t) - (2)\cdot y_2'(t)$

$$= u_1'(t)\cdot \left(y_1'(t)\cdot y_2(t) - y_1(t)\, y_2'(t)\right) = g(t)\cdot y_2(t).$$

$$u_1'(t) = \frac{g(t)\cdot y_2(t)}{-W(y_1, y_2)(t)}.$$

similarly $u_2'(t) = \dfrac{y_1(t)\cdot g(t)}{W(y_1, y_2)(t)}$